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Identification of Fuzzy Sets with a Class of Canonically Induced Random Sets and Some Applications

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Any random subset of a space X clearly determines the membership function of a fuzzy subset through its one point coverages. This paper shows, conversely, that any fuzzy subset A of X can always be identified with, in general, many random subsets S(A) of X with respect to one point coverages, i.e., simultaneously, for all $x \in X$, $\phi_A(x) = Pr(x \in S(A))$. In a related manner, it is shown that any fuzzy set can be uniformly closely approximated with respect to one point coverages by a random set having a finite number of outcomes.

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20. ABSTRACT (Continued)

In particular, the canonical mapping S_U, defined by $A \rightarrow S_U(A) \overset{\mathrm{df}}{=} \phi_A^{-1}([U,1])$, where A is any fuzzy subset of X and U is any uniformly distributed r.v. over [0, 1], produces such an identification. Moreover, S_U is an isomorphism from the collection of all fuzzy subsets onto a proper subcollection of all random subsets of X, with respect to many of the basic fuzzy set operations and corresponding ordinary set operations among random sets.

In addition, S_U, among all possible mappings from the class of all fuzzy subsets of X into the class of all random subsets of X which preserve one point coverages, induces both the maximal lower probability measure and the minimal upper probability measure in Dempster's sense on $P(X)$.

Applications of the results to fuzzy attribute reasoning in both military and general contexts are presented, emphasizing the close connection between fuzzy and random confidence sets.

Extensions of the above results to non-canonical mappings and multiple point coverage functions are also treated.

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CONTENTS

IDENTIFICATION OF FUZZY SETS WITH A CLASS OF CANONICALLY INDUCED RANDOM SETS AND SOME APPLICATIONS

INTRODUCTION

It is becoming increasingly more apparent that fuzzy set techniques can play an important role in formulating and solving military problems. For example, the recent paper of Watson et al. [1] makes basic contributions to the use of imprecise information which can occur typically in the decision problems facing a naval task force commander. Other applications of fuzzy set techniques such as fuzzy clustering [2,3] could well lead to more satisfactory target discrimination and correlation schemes than classical approaches. Fuzzy logic [4] might also play an important role in efficiently combining large quantities of military intelligence information which typically arise from disparate sources, and may in part consist of redundancies and vague specifications. (Examples 2 and 4 of the "Applications" section of this report illustrate this role of fuzzy logic.) On the other hand, procedures based on continually refined classical set theory and probabilistic-statistical methods still form the overwhelming majority of current technology used in a military context.

Since the inception of fuzzy set theory 15 years ago, a large number of papers have been produced in both the theoretical and applied areas. (See the 566 listings in Ref. 5, the 238 entries in Ref. 6, and a briefer listing by topics in Ref. 7.) However, little attention has been paid to determine what rigorous connections exist between fuzzy set theory and classical probability theory. At the same time, supporters of each school have been engaged in controversy involving approaches to the modeling of uncertainties. For example, see the comments of Stallings, a classical Bayesianist, vs Jain, a fuzzy set supporter [8-10]; Tribus, a standard probability theory supporter, vs Kandel and Zadeh, fuzzy set backers [5,11-13]; Zadeh's criticism [14] of Dempster's and Shafer's upper and lower probabilistic approaches [15-17] as well as his (Zadeh) development of possibility theory [18] as an alternative to classical probability theory. See also the introductory comments of Sugeno [19], stating unequivocally the impossibility of comparing fuzzy sets with probabilities.

These lively and extremely interesting discussions only serve to point up the need to answer the basic question:

What is the relationship between fuzzy set theory and classical probability theory?

This report demonstrates that indeed there is a direct connection between fuzzy set theory and random set theory, and that the fuzzy set approach to modeling of reality can be interpreted in equivalent probabilistic terms. On the other hand, this does not negate the various innovative fuzzy set *techniques* developed in the field, such as Sanchez' investigations and applications of solutions of relational equations to medical diagnoses [20] or the fuzzy reasoning approach to linguistically describable situations [4,13,18,21-23]. It appears that many problems are much better modeled initially in a fuzzy set context and then later, when so desired, described also in a probabilistic setting.

It is expected that the new relationships developed in this report will only serve to emphasize the mutual supportive roles probability theory and fuzzy set theory can play with respect to each other in future work.

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The basic problem can be stated as follows:

If S is any random subset of a space X, then clearly S determines a naturally corresponding fuzzy subset A(S) of X which is equivalent to S under all one point coverages, i.e.,

$$\phi_{A(S)}(x) \overset{df}{=} \Pr(x \in S) \quad , \tag{1.1}$$

for all $x \in X$.

However, the converse of this situation is not all obvious: If A is any given fuzzy subset of X, does there exist a random set S(A), which is equivalent to A under one point coverages, i.e.,

$$\Pr(x \in S(A)) = \phi_A(x) \quad , \tag{1.2}$$

for all $x \in X$?

THE BASIC CANONICAL MAPPING

The following definitions and notations are assumed throughout:

X is an arbitrary fixed space, F is the collection of all fuzzy subsets A of X, each A having typical membership function $\phi_A : X \to [0,1]$, and R is the collection of all random subsets S of X. Denote the collection of all ordinary subsets of X by $P(X)$.

For any fuzzy subset A of X, define the multi-valued mapping

$$\Gamma_A : [0,1] \to P(X) \quad , \tag{2.1}$$

where, for any $x \in [0, 1]$

$$\Gamma_A(x) \overset{df}{=} \phi_A^{-1}([x,1]) \quad . \tag{2.2}$$

If U is any random variable uniformly distributed over [0,1]—corresponding to unit interval probability space

$$I \overset{df}{=} ([0,1], B_1, \mathrm{vol}_1) \quad , \tag{2.2'}$$

where B_1 is the σ-ring of all Borel, or more generally, Lebesgue measurable subsets of [0,1] and vol_1 is Lebesgue measure—then define the mapping

$$S_U : F \to R \quad , \tag{2.3}$$

where, for any $A \in F$,

$$S_U(A) \overset{df}{=} \Gamma_A(U) = \phi_A^{-1}([U, 1]) \quad . \tag{2.4}$$

Also, define

$$\mathrm{rng}(\Gamma_A) \overset{df}{=} \{\Gamma_A(x) \mid x \in [0,1]\} \quad , \tag{2.5}$$

$$\Gamma_A(B_1) \overset{df}{=} \{\Gamma_A(B) \mid B \in B_1\} \overset{df}{=} \{\{\Gamma_A(x) \mid x \in B\} \mid B \in B_1\} \quad , \tag{2.6}$$

and for any $A \in \Gamma_A(B_1)$, i.e., $A = \Gamma_A(B)$, for some $B \in B_1$,

$$\mu_{\Gamma_A}(A) \overset{df}{=} \mathrm{vol}_1(\Gamma_A^{-1}(A)) \quad . \tag{2.7}$$

Theorem 1

For any fuzzy subset A of X with rng $(\phi_A) \in \mathcal{B}_1$:

1. Γ_A is a measurable mapping from probability space $\mathcal{I}$ to probability space

$$\mathcal{S}_A \overset{\text{df}}{=} (\text{rng}\,(\Gamma_A), \Gamma_A(\mathcal{B}_1), \mu_{\Gamma_A}) \; . \tag{2.8}$$

$\mathcal{S}_A$ can be considered to be the probability space corresponding to random set S_U (A) , since, for all $\mathcal{A} \in \Gamma_A(\mathcal{B}_1)$,

$$\mu_{\Gamma_A}(\mathcal{A}) = \Pr\,(S_U\,(A) \in \mathcal{A}) \; . \tag{2.9}$$

2. A and S_U (A) are equivalent with respect to all one point coverages:

For all $x \in X$,

$$\phi_A\,(x) = \Pr\,(S_U\,(A) \in \mathcal{C}_{\{x\}}) = \Pr\,(x \in S_U\,(A)) \; , \tag{2.10}$$

where

$$\mathcal{C}_{\{x\}} \overset{\text{df}}{=} \{C | C \in \text{rng}(\Gamma_A)\ \&\ x \in C\} = \Gamma_A\,([0, \phi_A(x)]) \in \Gamma_A(\mathcal{B}_1) \; . \tag{2.11}$$

Proof 1:

Let $\mathcal{A} = \Gamma_A(B) \in \Gamma_A(\mathcal{B}_1)$ be arbitrary, for some $B \in \mathcal{B}_1$ (not a unique representation, in general). For any $y \in [0, 1]$, define $I_y \overset{\text{df}}{=} \{x | x \in [0,1]\ \&\ \phi_A^{-1}\,([x,1]) = \phi_A^{-1}\,([y,1])\}$.

Note, for $x \neq y$, $\phi_A^{-1}\,([x,1]) = \phi_A^{-1}\,([y,1])$ implies $\phi_A^{-1}\,([\min\,(x, y), \max\,(x, y))) = \varnothing$. Then

$$\Gamma_A^{-1}\,(\mathcal{A}) = \bigcup_{y \in B} I_y = (\bigcup_{y \in B \cap J(A)} I_y) \cup (B \dashv J(A)) \; , \tag{2.7$'$}$$

where

$$J(A) \overset{\text{df}}{=} \{y | y \in [0,1]\ \&\ I_y \text{ is a left half closed nondegenerate interval containing } y\} \; ;$$

$[0,1] \dashv J(A) = \{y | y \in [0,1]\ \&\ I_y = \{y\}\} = \{y | y \in [0, 1]\ \&$ for any $\epsilon > 0$, there exist $x_1, x_2 \in$ rng (ϕ_A) such that $x_1 < y \leqslant x_2\ \&\ |x_2 - x_1| < \epsilon\}$.

Since rng $(\phi_A) \in \mathcal{B}_1$, then $J(A) \in \mathcal{B}_1$, and thus $B \dashv J(A) \in \mathcal{B}_1$. In addition, $\bigcup_{y \in B \cap J(A)} I_y$ is actually a disjoint, and hence at most countably infinite, union (since the I_y's are intervals) $\bigcup_{y \in K(A,B)} I_y \in \mathcal{B}_1$, where $K(A,B) \subseteq B \cap J(A)$.

Thus, Eq. (2.7$'$) becomes

$$\Gamma_A^{-1}\,(\mathcal{A}) = \bigcup_{y \in K(A,B)} I_y \cup (B \dashv J(A)) \in \mathcal{B}_1 \; . \tag{2.7$''$}$$

Next, consider $\Gamma_A(\mathcal{B}_1)$. If $B_1, B_2, \ldots, \in \mathcal{B}_1$, clearly, $\bigcup_{j=1}^{\infty} \Gamma_A(B_j) = \Gamma_A(\bigcup_{j=1}^{\infty} B_j) \in \Gamma_A\,(\mathcal{B}_1)$.

However, for intersections

$$\bigcap_{j=1}^{\infty} \Gamma_A(B_j) = \Gamma_A\,(\Gamma_A^{-1}(\bigcap_{j=1}^{\infty} \Gamma_A(B_j))) = \Gamma_A(\bigcap_{j=1}^{\infty} \Gamma_A^{-1}\,(\Gamma_A(B_j))) \in \Gamma_A(\mathcal{B}_1) \quad ,$$

since Eq. (2.7″) implies each $\Gamma_A^{-1}\,(\Gamma_A(B_j)) \in \mathcal{B}_1$, and hence $\bigcap_{j=1}^{\infty} \Gamma_A^{-1}\,(\Gamma_A(B_j)) \in \mathcal{B}_1$.

Similarly, for any $B \in \mathcal{B}_1$,

$$\mathrm{rng}(\Gamma_A) \dashv \Gamma_A(B) = \Gamma_A\,(\Gamma_A^{-1}(\mathrm{rng}(\Gamma_A) \dashv \Gamma_A(B))) = \Gamma_A\,(\,[0,1] \dashv \Gamma_A^{-1}\,(\Gamma_A(B)))$$

with Eq. (2.7″) implying $\mathrm{rng}\,(\Gamma_A) \dashv \Gamma_A^{-1}(\Gamma_A(B)) \in \mathcal{B}_1$, and hence $\mathrm{rng}\,(\Gamma_A) \dashv \Gamma_A(B) \in \Gamma_A(\mathcal{B}_1)$.

Thus, $\Gamma_A\,(\mathcal{B}_1)$ is a σ-ring, and therefore Eq. (2.7″) implies Γ_A is measurable. The induced measure μ_{Γ_A} is from Eqs. (2.7″) and (2.7),

$$\mu_{\Gamma_A}(\mathcal{A}) = \sum_{y \in K(A,B)} \mathrm{vol}_1\,(I_y) + \mathrm{vol}_1(B \dashv J(A)) \quad , \tag{2.7‴}$$

where $\mathrm{vol}\,(I_y) = \sup(I_y) - \inf\,(I_y)$, for all y.

Proof 2:

Follows immediately from Eq. (2.4).

□

From now on, the condition $\mathrm{rng}\,(\phi_A) \in \mathcal{B}_1$ will be assumed, where required.

Theorem 2

Any fuzzy subset A of space X can be uniformly approximated arbitrarily close w.r.t. one point coverages by a random set which has a finite number of possible set values:

For any integer $n \geqslant 2$, let U_n be a r.v. uniformly distributed over the discrete set

$$D_n \overset{\mathrm{df}}{=} \{i/(n-1) \mid i = 0,1,.., n-1\} \quad , \tag{2.12}$$

and define, analogous to the continuous case,

$$\underset{U_n;n}{S}\,(A) \overset{\mathrm{df}}{=} \phi_A^{-1}\,([U_n,1]) \quad , \tag{2.13}$$

Then,

$$\sup_{x \in X} |\,\Pr\,(x \in \underset{U_n;n}{S}\,(A)) - \phi_A(X)\,| \leqslant 1/n \quad . \tag{2.14}$$

Proof:

Partition X into disjoint sets of the form $\phi_A^{-1}\,([i/(n-1),\;(i+1)/(n-1)])$ and consider the maximum of suprema over those sets.

□

Remarks

Let A be any fuzzy subset of X:

Dempster's lower and upper probability measures [14-17,24] induced on *P* (X) by the multi-valued mapping Γ_A, become here

$$Pr_*(C) = Pr(\emptyset \neq S_U(A) \subseteq C)/Pr(\emptyset \neq S_U(A)) , \tag{2.15}$$

$$Pr^*(C) = Pr(\emptyset \neq S_U(A) \cap C)/Pr(\emptyset \neq S_U(A)) , \tag{2.16}$$

for any $C \in P(X)$.

Assuming now for simplicity that there is at least one $x \in X$ such that $\phi_A(x) = 1$, Eqs. (2.15) and (2.16) simplify accordingly. Then for *any* random subset S of X which is equivalent to A w.r.t. one point coverages, for all $C \in P(X)$:

$$\begin{aligned} Pr(\emptyset \neq C \cap S) &\geq \sup_{x \in C} Pr(x \in S) = \sup_{x \in C} Pr(x \in S_U(A)) \\ &= \sup_{x \in C} \phi_A(x) \overset{df}{=} \text{Poss (fuzzy variable A is `in' C)} = Pr^*(C) \\ &= Pr(\emptyset \neq C \cap S_U(A)) , \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} Pr(C \subseteq S) &\leq \inf_{x \in C} Pr(x \in S) = \inf_{x \in C} Pr(x \in S_U(A)) \\ &= \inf_{x \in C} \phi_A(x) \overset{df}{=} \text{Cert (fuzzy variable A is `in' C)} = Pr_*(C) \\ &= Pr(C \subseteq S_U(A)) . \end{aligned} \tag{2.18}$$

Thus, S_U achieves, among all possible maps from *F* to *R* which preserve one point coverages, dually (1), a maximum lower probability Dempster measure, i.e., maximizes all (conjunctive) multiple point coverages and (2), a minimum upper probability Dempster measure, i.e., minimizes all disjunctive multiple point coverages.

□

Theorem 3

The mapping S_U between all fuzzy and random subsets of a space has isomorphic or isomorphic-like properties with respect to a number of fuzzy set operations and corresponding ordinary random set ones:

(i) For any A, B, fuzzy subsets of X

$$S_U(A \cup B) = S_U(A) \cup S_U(B) \tag{2.19}$$

$$S_U(A \cap B) = S_U(A) \cap S_U(B) \tag{2.20}$$

$$S_U(X \dashv A) = X \dashv S_{1-U}(A) \tag{2.21}$$

$$S_U(A \Rightarrow B) = (S_{1-U}(A) \Rightarrow S_U(B)) \tag{2.22}$$

(ii) For any fuzzy subset A of X, fuzzy subset B of Y, X, Y given spaces,

$$S_U(A \times B) = S_U(A) \times S_U(B) , \tag{2.23}$$

i.e., denoting $proj_1$ as the projection onto X-space, etc.,

$$proj_1 (S_U (A \times B) = S_U (A) \quad , \quad proj_2 (S_U (A \times B)) = S_U (B) \ .$$

(iii) For any A, B, fuzzy subsets of X,

$$A \subseteq B \text{ iff } (S_U (A) \subseteq S_U (B) \ , \text{for all outcomes } U \in [0,1]) \ . \tag{2.24}$$

(iv) Let A be a fuzzy subset of X and B a fuzzy subset of Z, and $f : X \overset{\text{into}}{\rightarrow} Z$, an ordinary function between X and Z, given spaces. Then:

$$S_U (f(A)) = f(S_U (A)) \quad \text{and} \quad S_U (f^{-1} (B)) = f^{-1} (S_U (B)) \ . \tag{2.25}$$

(v) For any fuzzy subset A of X × Y, X , Y given spaces, for all $x \in X$, $y \in Y$

$$Pr (x \in proj_1(S_U (A))) \geqslant Pr (x \in S_U (proj_1 (A))) = \phi_{proj_1(A)} (x) \tag{2.26}$$

$$\geqslant \phi_A (x, y) = Pr (x, y \in S_U (A)) \ .$$

Proofs and Remarks:

Recall [7,25,26], the basic fuzzy set operations

$$\phi_{A \cup B} (x) \overset{df}{=} \max (\phi_A (x), \phi_B (x)) \ ,$$
$$\phi_{A \cap B} (x) \overset{df}{=} \min (\phi_A (x), \phi_B (x)) \ ,$$
$$\phi_{X \dashv A}(x) \overset{df}{=} 1 - \phi_A (x) \ ,$$
$$\phi_{A \times B} (x, y) \overset{df}{=} \min (\phi_A (x), \phi_B (y)) \ ,$$
$$A \subseteq B \text{ iff } \phi_A (x) \leqslant \phi_B (x), \text{ for all } x \ ,$$
$$\phi_{f(A)} (y) \overset{df}{=} \sup_{x \in f^{-1}(y)} \phi_A (x) \ ,$$
$$\phi^{(x)}_{f^{-1}(B)} \overset{df}{=} \phi_B (f(x)) \ ,$$
$$\phi_{A \Rightarrow B} (x) = \phi_{(X \dashv A) \cup B} (x) = \max (1 - \phi_A (x), \ \phi_B (x)) \ ,$$
$$\phi_{proj_1(C)} (x) = \sup_{y \in Y} \phi_C (x, y) \ ,$$

where C is a fuzzy subset of X × Y , $x \in X$.

The random set operations are standard, noting it is the *same* outcome of U that governs both $S_U (A)$ and $S_U (B)$ in the R.H.S. of Eqs. (2.19), (2.20), (2.23), and (2.24). Note that r.v. U is replaced in the R.H.S. of Eq. (2.21) by the identically distributed but distinct $1 - U$. Similarly for the R.H.S. of Eq. (2.22), where '$\Rightarrow$' is defined between any two ordinary sets S and $T \subseteq X$ as $(X \dashv S) \cup T$. $proj_1$ as a random set operation in the R.H.S. of Eq. (2.26) is defined in the usual manner :

$$x \in proj_1(S) \text{ iff } (x,y) \in S \ ,$$

for at least some $y \in Y$, where S is any outcome of a random subset of X × Y; $x \in X$.

All proofs then follow by straightforward use of the definitions.

□

Additional Remarks

1. Let T be any random subset of space X and define fuzzy set A(T) as in Eq. (1.1). Then although S_U (A(T)) and T (and A(T)) are equivalent w.r.t. all one point coverages, in general the two random sets are quite distinct. Note also that $A(S_U(A(T))) = A(T)$.

To illustrate how different S_U (A(T)) and T may be, suppose $T = [W - a/2, W + a/2]$ is a random interval of $\mathbb{R}$, where $a > 0$ is a constant and W is a random variable having p.d.f. h which is everywhere positive and continuous over $\mathbb{R}$. Then, for any two fixed points $x, y \in \mathbb{R}$ with $|x - y| \geqslant a$,

$$0 = \Pr(\{x,y\} \subseteq T) < \min\left[\int_{x-a/2}^{x+a/2} h(t)dt \;,\; \int_{y-a/2}^{y+a/2} h(t)dt\right]$$
$$= \min(\Pr(x \in T) \;,\; \Pr(y \in T)) = \Pr(\{x,y\} \subseteq S_U(A)) \;. \qquad (2.27)$$

(See also Eq. (2.18).)

2. If A is an ordinary subset of X,

$$1 = \Pr(S_U(A) = \phi_A^{-1}(1) = A) \;.$$

(The probability qualification is needed, since $\Pr(U = 0) = 0$, but $S_A(0) = X$.)

3. If $X \subseteq \mathbb{R}$ and ϕ_A is monotone increasing over X, then $S_U(A) = [\phi_A^{-1}(U), 1]$.

4. If, unlike the random sets on the right-hand side (R.H.S.) of Eqs. (2.19), (2.20), and (2.23), two *statistically independent* uniform r.v.'s U and V, say, are present, then, for any fuzzy subsets A and B of X and all $x \in X$,

$$\Pr(x \in S_U(A) \cup S_V(B)) = \phi_{A \oplus B}(x) \overset{\mathrm{df}}{=} \phi_A(x) + \phi_B(x) - \phi_A(x) \cdot \phi_B(x)$$
$$\geqslant \max(\phi_A(x), \phi_B(x)) = \Pr(x \in S_U(A) \cup S_U(B)) = \Pr(x \in S_U(A \cup B)) \;. \qquad (2.28)$$

Similarly,

$$\Pr(x \in S_U(A) \cap S_V(B)) = \phi_{A \cdot B}(x) \overset{\mathrm{df}}{=} \phi_A(x) \cdot \phi_B(x) \leqslant \min(\phi_A(x), \phi_B(x))$$
$$= \Pr(x \in S_U(A) \cap S_U(B)) = \Pr(x \in S_U(A \cap B)) \;. \qquad (2.29)$$

Strict inequality, in general, holds in Eqs. (2.28) and (2.29).

Note that both $S_U(A) \cup S_V(B)$ and $S_U(A \oplus B)$ are equivalent (to $A \oplus B$) w.r.t. one point coverages, but are quite distinct in structure.

Similarly, $S_U(A) \cap S_V(B)$ and $S_U(A \cdot B)$ are equivalent w.r.t. one point coverages (to $A \cdot B$), but are also different in form.

5. The definition of a probability measure of a fuzzy subset A of ordinary probability space (X, B, μ) having a corresponding random variable Y which is assumed statistically independent of U, becomes here (see, e.g. Ref. 7, p. 33)

$$\Pr(A) \overset{df}{=} E(\phi_A(Y)) = \Pr(Y \in S_U(A)) = \int_{x \in X} \Pr(x \in S_U(A))\, d\mu(x)$$

$$= \int_{u=0}^{1} \Pr(Y \in S_u(A))\, du \quad . \tag{2.30}$$

6. For any fuzzy subset A of X × Y, it is always true

$$A \subseteq \mathrm{proj}_1(A) \times \mathrm{proj}_2(A) \quad , \tag{2.31}$$

and hence by Eqs. (2.23) and (2.24),

$$S_U(A) \subseteq S_U(\mathrm{proj}_1(A)) \times S_U(\mathrm{proj}_2(A)) \quad . \tag{2.32}$$

If A is such that equality holds in Eq. (2.31), then $\mathrm{proj}_1(A)$ and $\mathrm{proj}_2(A)$ are said to be *non-interactive* w.r.t. A , which is clearly equivalent, by Eqs. (2.23) and (2.24) to equality holding in Eq. (2.32). Thus, in this notation A and B are non-interactive w.r.t. A × B.

7. Note that for the A ( ) mapping as in Eq. (1.1), the analogue of Theorem 3 is a restatement of the usual laws of probability theory. For examples, if S and T are any statistically independent random subsets of X

$$A(S \cup T) = A(S) \oplus A(T) \quad , \tag{2.33}$$

$$A(S \cap T) = A(S) \cdot A(T) \quad . \tag{2.34}$$

It is always true that

$$A(X \dashv S) = X \dashv A(S) \quad , \tag{2.35}$$

8. A *linguistic variable* A (see, e.g. Ref. 4, section 3) in its simplist form can be considered as a generalized fuzzy set where the range of the membership function of A consists of membership functions of ordinary fuzzy sets:

$$\phi_A : X \to [0, 1]^X \quad . \tag{2.36}$$

Using Theorem 1, the probabilistic interpretation becomes, for each x and w, ∈ X where A_w is some fuzzy subset of X

$$\phi_A(w) = \phi_{A_w} = (\phi_{A_w}(x))_{x \in X} \quad ; \tag{2.37}$$

$$\phi_{A_w}(x) = \Pr(x \in S_U(A_w)) \quad . \tag{2.38}$$

□

SOME APPLICATIONS OF THE CANONICAL MAPPING TO FUZZY LOGIC

In this section, fuzzy reasoning and classical logical reasoning are shown to be related. In particular, if the premise of an argument consists of conjunctions of fuzzy and/or probabilistic inequalities, equivalent fuzzy and random confidence sets may be derived for the unknown variable in question. Indeed, it appears that no real information loss—nor any real change in structure—occurs if the *entire* reasoning is carried out in a fuzzy logic (or set) context, and then translated in terms of an equivalent probability statement, if so desired. (For basic background, illustrative examples and definitions used in fuzzy reasoning, see Refs. 4, 14, 21, 22 and 23.)

Suppose $V = (v_1, \ldots, v_m) \in \mathbf{R}^m$ or $\Omega_1 \times \ldots \times \Omega_m$ for more abstract sets Ω_i, $i = 1, \ldots, m$, where each v_i represents some characteristic: dimensional, such as age in years of Mr. C, height in inches of building D, or nondimensional, such as amount of membership in a given fuzzy set such as tallness, goodness, etc.

V is not considered a random quantity, but rather an unknown quantity, with values restricted by n propositional functions $A_1, \ldots, A_n$, which are identified as fuzzy subsets of $\mathbf{R}^m$, and by n lower bound membership levels $\alpha_1, \ldots, \alpha_n \in [0,1]$ which may or may not be specified throughout the problem. Each propositional function, in turn, may be internally composed of logical relations between other more primative ones. In that case, it is natural to identify the logical relations 'and', 'or', 'not', 'if then', with the fuzzy operations '$\cap$', '$\cup$', '$\mathbf{R}^m \dashv ()$', '$\twoheadrightarrow$', respectively. For purposes of simplicity, it will be assumed that all of the restrictions on V hold conjunctively.

In addition, suppose $T_1, \ldots, T_r$ are r random subsets of $\mathbf{R}^m$, r possibly vacuous, individually supplying information about V at confidence levels $\beta_1, \ldots, \beta_r$, respectively, with the joint distribution of the T_j's not necessarily specified. (For both the propositional and probabilistic statements, any information concerning subvariables of V is always formally expressible in terms of all of V.)

Then, the premise of the augument concerning V can be stated as:

$$(\phi_{A_1}(V) \geqslant \alpha_1) \,\&\, (\phi_{A_2}(V) \geqslant \alpha_2) \,\&\, \ldots \,\&\, (\phi_{A_n}(V) \geqslant \alpha_n)$$
$$\&(\Pr(V \in T_1) \geqslant \beta_1) \,\&\, \ldots \,\&\, (\Pr(V \in T_r) \geqslant \beta_r). \tag{3.1}$$

Note that the canonical mapping in Eq. (2.4) could be applied separately to each fuzzy set A_i in Eq. (3.1) so that $(\phi_{A_i}(V) \geqslant \alpha_i)$ is replaced by $(\Pr(V \in S_{U_i}(A_i)) \geqslant \alpha_i)$, where $U_1, \ldots, U_n$ are uniformly distributed over $[0,1]$, but with joint distribution of the U_i's (and T_j's) not specified. Thus, e.g., it is possible to choose $U_1 = \cdots = U_n$ and to be stat. indep. of the T_j's or to let all U_i's and T_j's to be stat. indep.

In practice, the distribution of each T_j is often known only in the conditional form $\Pr(V \in T_j \mid V)$, where V also plays the role of an unknown parameter, and where a sample outcome, say $\hat{T}_j$ has been observed from T_j. In this case, in order to have all one point coverages well defined, it appears reasonable (via a fiducial type of argument) to redefine T_j so that Eq. (1.1) becomes

$$\Pr(V \in T_j) = \phi_{A(T_j)}(V) \begin{cases} \geqslant \beta_j, & \text{for all } V \in \hat{T}_j \\ \leqslant 1 - \beta_j, & \text{for all } V \notin \hat{T}_j \end{cases} \,. \tag{3.2}$$

Regardless of the joint distribution of the U_i's *and* T_j's, for all V satisfying Eq. (3.1),

$$\alpha \stackrel{\text{df}}{=} \min(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_r)$$
$$\leqslant \min(\phi_{A_1}(V), \ldots, \phi_{A_n}(V), \phi_{A(T_1)}(V), \ldots, \phi_{A(T_r)}(V))$$
$$= \phi_B(V) = \Pr(V \in S_U(B)), \tag{3.3}$$

which is formally the same as choosing $U_1 = \cdots = U_n = U$, *replacing the '&' 's in Eq. (3.1) by fuzzy intersections and forming* B, where

$$B \stackrel{\text{df}}{=} A_1 \cap \cdots \cap A_n \cap A(T_1) \cap \cdots \cap A(T_r) \,. \tag{3.4}$$

Note, from Eq. (2.20),

$$S_U(B) = S_U(A_1) \cap \cdots \cap S_U(A_n) \cap S_U(A(T_1)) \cap \cdots \cap S_U(A(T_r)) \,. \tag{3.5}$$

The expressions on the R.H.S. of Eq. (3.3) consist of an equivalent (w.r.t. all one point coverages) *α-level fuzzy confidence set* B, and a *random confidence set* S_U (B) for all V satisfying Eq. (3.1).

It should be remarked that even if all of the U_i's and T_j's mentioned above were assumed statistically independent, the resulting natural intersection γ-level random confidence set (for all V satisfying Eq. (3.1)) determined by Q in Eqs. (3.3′) and (3.5′) is not necessarily preferable to the α-level one in Eq. (3.3) determined by S_U (B), where

$$\gamma \overset{\mathrm{df}}{=} \alpha_1 \cdots \alpha_n \cdot \beta_1 \cdots \beta_r \quad \leqslant \Pr(V \in Q) \ , \tag{3.3'}$$

where

$$Q \overset{\mathrm{df}}{=} S_{U_1}(A_1) \cap \cdots \cap S_{U_n}(A_n) \cap T_1 \cap \cdots \cap T_r \ . \tag{3.5'}$$

The above remark can be justified by first noting that γ is not as tight a lower bound as α. Furthermore, when $\alpha_1 = \cdots = \alpha_r = \beta_1 = \cdots = \beta_r$, if V is such that Eq. (3.3) holds, then V satisfies Eq. (3.3′), and clearly Eq. (3.3) is preferable to Eq. (3.3′). On the other hand, when, say, $\alpha_1, \ldots, \alpha_n$, $\beta_2, \ldots, \beta_r$ are close to unity, with β_1 arbitrary, it follows that if V is such that Eq. (3.3′) holds, then Eq. (3.3) holds approximately and thus Eq. (3.3′) is preferable to Eq. (3.3).

Often, it is desired to obtain confidences about one or more components of V. For example, applying Eq. (2.26) to Eq. (3.3), confidence sets for v_1 can be obtained from the premise about V in Eq. (3.1):

For all v_1 satisfying Eqs. (3.1), (3.2)

$$\alpha \leqslant \phi_C(v_1) \quad = \Pr(v_1 \in S_U(C)) \quad \leqslant \Pr(v_1 \in \mathrm{proj}_1(S_U(B))) \ , \tag{3.6}$$

where

$$C \overset{\mathrm{df}}{=} \mathrm{proj}_1(B) \ , \tag{3.7}$$

and B is given in Eq. (3.4).

Thus, Eq. (3.6) yields C, an α-level fuzzy confidence set for v_1, an equivalent α-level random confidence set S_U (C) for v_1, and a somewhat more complicated (non-equivalent) α-level random confidence set, proj_1 (S_U (B)) for v_1, for all v_1 (and V) satisfying Eq. (3.1).

A brief example of applications to a disjunctive premise:

$$(\phi_{A_1}(V) \geqslant \alpha_1) \text{ or } (\phi_{A_2}(V) \geqslant \alpha_2) \tag{3.8}$$

implies

$$\begin{aligned} \alpha \overset{\mathrm{df}}{=} \min(\alpha_1, \alpha_2) \quad &\leqslant \max(\phi_{A_1}(V), \phi_{A_2}(V)) \quad = \phi_{A_1 \cup A_2}(V) \\ &= \Pr(V \in S_U(A_1 \cup A_2)) = \Pr(V \in S_U(A_1) \cup S_U(A_2)) \ . \end{aligned} \tag{3.9}$$

Logical relations forming the premise other than conjunctive or disjunctive ones could be handled by either translating in terms of possible negations or affirmations of the above connectives, or perhaps more satisfactorally modeled directly in terms of fuzzy relations, as in Example 4 for ‘if ( ) then ( ) ’.

The following four examples illustrate the previous discussions.

Example 1. (See Zadeh [23], example 1, p. 23)

Premise : v_1 is a small positive real number; v_2 is a positive real number such that v_1 and v_2 are approximately equal; and v_3 is a positive real number that is much larger than both v_1 and v_2.

Conclusion : How large is v_3?

Define

$$V = (v_1, v_2, v_3) \in \mathbb{R}^3 \ , \tag{3.10}$$

and fuzzy sets

$$A_1 \overset{\mathrm{df}}{=} \text{small} \tag{3.11}$$

$$A_2 \overset{\mathrm{df}}{=} \text{approximately equal} \ , \tag{3.12}$$

$$A_3 \overset{\mathrm{df}}{=} \text{much larger than (for third component w.r.t. first two)} \ , \tag{3.13}$$

where for all $V = (v_1, v_2, v_3) \in \mathbb{R}^3$:

$$\phi_{A_1}(V) = \phi_{A_1}(v_1) \text{ is a monotone decreasing (from 1 to 0) function in } v_1 \text{ over } \mathbb{R}^+ \ ; \tag{3.14}$$

$$\phi_{A_2}(V) = \phi_{A_2}(v_1, v_2) = \phi_{A_1}(|v_1 - v_2|) \ ; \tag{3.15}$$

i.e., v_1 is approximately equal to v_2 iff $|v_1 - v_2|$ is small;

$$\phi_{A_3}(V) = \begin{cases} \min\left((1 - \phi_{A_1}(v_3 - v_1))^4,\ (1 - \phi_{A_1}(v_3 - v_2))^4\right) \ , \\ \text{iff } v_3 \geqslant v_1, v_2 \\ \text{-----------------} \\ 0 \ , \\ \text{iff } v_3 < v_1 \text{ or } v_3 < v_2 \ , \end{cases} \tag{3.16}$$

i.e., v_3 is much greater than both v_1 and v_2 iff (($v_3 - v_2$ is very^2 not small) & ($v_3 - v_1$ is very^2 not small)) , where for any fuzzy set membership function ϕ_A , $\phi_{\mathrm{not}(A)} = 1 - \phi_A$, $\phi_{\mathrm{very}(A)} = \phi_A^2$, $\phi_{\mathrm{very}^2(A)} = \phi_A^4$, etc.

Then the premise becomes here

$$(\phi_{A_1}(V) \geqslant \alpha_1) \ \&\ (\phi_{A_2}(V) \geqslant \alpha_2) \ \&\ (\phi_{A_3}(v) \geqslant \alpha_3) \ , \tag{3.17}$$

for some unspecified $1 \geqslant \alpha_1, \alpha_2, \alpha_3 \geqslant 0$.

Equation (3.6) then implies the conclusion

$$\alpha \overset{\mathrm{df}}{=} \min(\alpha_1,\alpha_2,\alpha_3) \leqslant \phi_C(v_3) = \Pr(v_3 \in S_U(C)) \leqslant \Pr(v_3 \in \mathrm{proj}_3(S_U(B))) \ , \tag{3.18}$$

where

$$B \overset{\mathrm{df}}{=} A_1 \cap A_2 \cap A_3 \tag{3.19}$$

and

$$C \overset{\mathrm{df}}{=} \mathrm{proj}_3(B) \ . \tag{3.20}$$

For all $v_3 \in \mathbb{R}^+$,

$$\phi_C(v_3) = \sup_{\substack{\text{over all} \\ v_1, v_2 \in \mathbb{R}^+}} \phi_B(V) = \sup_{v_1, v_2 \in \mathbb{R}^+} \min(\phi_{A_1}(v_1), \phi_{A_2}(v_2), \phi_{A_3}(v_3)) \tag{3.21}$$

$$= (1 - \phi_{A_1}(v_3))^4 \quad ,$$

by substituting from Eqs. (3.14), (3.15), and (3.16) and graphical inspection.

Thus, C can be interpreted as

$$C = \text{very}^2 \text{ not small} = \text{very large} \quad . \tag{3.22}$$

Eqs. (3.18) and (3.21) imply that

$$v_3 \geqslant \phi_{A_1}^{-1}(1 - \alpha^{1/4}) \quad , \tag{3.23}$$

and

$$S_U(C) = [\phi_{A_1}^{-1}(1 - U^{1/4}), 1] \quad . \tag{3.24}$$

Example 2. (See Zadeh [13], problem 3, page 2)

Premise:

Source 1: Most ships spotted in the surveillance area of interest have tall masts.

Source 2: It is known from previous experience that most of the tall masted ships are enemy ships.

Conclusion 1: How many ships in the area are enemy ones?

Conclusion 2: How many ships in the area are tall masted enemy ships?

Define

$$G \overset{\text{df}}{=} \{g_1, \ldots, g_n\} = \text{set of all ships in area} \ , \tag{3.25}$$

$$n \overset{\text{df}}{=} \text{card}(G) \quad \text{(estimated)} \ , \tag{3.26}$$

and fuzzy sets

$$A_1 \overset{\text{df}}{=} \text{enemy} \ , \tag{3.27}$$

$$A_2 \overset{\text{df}}{=} \text{tall masted} \ , \tag{3.28}$$

where ϕ_{A_2}: [0,1] → [0,1] is a monotone increasing function, where if $g \in G$,

$$\text{ht}(g) \overset{\text{df}}{=} \text{measure of mast height in feet of } g \ , \tag{3.29}$$

and $\phi_{A_2}(\text{ht}(g))$ is a measure of mast tallness of g .

$$\phi_{A_1}: G \to [0,1] \tag{3.30}$$

unknown may be a fuzzy or ordinary set,

$$\Omega_1 = \cdots = \Omega_{2n+1} = [0,1] \ , \tag{3.31}$$

$$V = (v_1, \ldots, v_{2n}, v_{2n+1}) \ , \tag{3.32}$$

$v_{2n+1} \in [0, 1]$,

$v_i \stackrel{df}{=} \phi_{A_1}(g_i)$, $i = 1, .., n$, (3.33)

$v_{n+i} \stackrel{df}{=} \phi_{A_2}(g_i)$, $i = 1, \ldots, n$, (3.34)

$f(v_1, .., v_n) \stackrel{df}{=} (1/n) \sum_{i=1}^{n} v_i$ = % of ships in area that are enemy , (3.35)

$h(v_{n+1}, .., v_{2n}) \stackrel{df}{=} (1/n) \sum_{i=1}^{n} v_{n+i}$ = % of ships in area that are tall masted , (3.36)

$j(v_1, .., v_{2n}) \stackrel{df}{=} (1 / \sum_{i=1}^{n} v_{n+i}) \sum_{i=1}^{n} v_i v_{n+i}$

= % of tall masted ships in area that are also enemy ones , (3.37)

$k(v_1, .., v_{2n}) \stackrel{df}{=} (1/n) \cdot \sum_{i=1}^{n} v_i \cdot v_{n+i}$ = % of ships in the area that are tall masted enemy ones . (3.38)

(Note that for mathematical convenience, the possibly more appropriate expression $\min(v_i, v_{n+i})$ has been replaced by $v_i \cdot v_{n+i}$.)

Also, define the fuzzy sets

$A_3 \stackrel{df}{=}$ most , (3.39)

where $\phi_{A_3}: [0, 1] \to [0, 1]$ is a monotone increasing function (sharply so, after 0.5),

$A_4 \stackrel{df}{=}$ ordinary function f , (3.40)

where $\phi_{A_4}: \Omega_1 \times \cdots \times \Omega_n \times \Omega_{2n+1} \to [0,1]$ is such that for any $v_i \in \Omega_i = [0,1]$, $i = 1, .., n$, and $x \in \Omega_{2n+1} = [0,1]$,

$$\phi_{A_4}(v_1, \ldots, v_n, x) = \begin{cases} 1 & \text{iff } x = f(v_1, .., v_n) \\ 0 & \text{iff } x \neq f(v_1, .., v_n) \end{cases} , \quad (3.41)$$

$A_5 \stackrel{df}{=}$ ordinary function j , (3.42)

where $\phi_{A_5}: \Omega_1 \times \cdots \times \Omega_{2n} \times \Omega_{2n+1} \to [0,1]$ is such that for any $v_i \in \Omega_i = [0, 1]$, $i = 1, .., 2n$, and $y \in \Omega_{2n+1} = [0, 1]$,

$$\phi_{A_5}(v_1, \ldots, v_{2n}, y) = \begin{cases} 1 & \text{iff } y = k(v_1, .., v_{2n}) \\ 0 & \text{iff } y \neq k(v_1, .., v_{2n}) \end{cases} . \quad (3.43)$$

Then the premise here becomes

$$(\phi_{A_3}(h(v_{n+1}, \ldots, v_{2n})) \geqslant \alpha_1) \,\&\, (\phi_{A_3}(j(v_1, \ldots, v_{2n})) \geqslant \alpha_2) \,\&\, (\phi_{A_4}(v_1, \ldots, v_n, x) = 1)$$
$$\&\, (\phi_{A_5}(v_1, \ldots, v_{2n}, y) = 1) . \quad (3.44)$$

For conclusion 1:

Equation (3.6) implies

$$\alpha \overset{df}{=} \min(\alpha_1, \alpha_2) \leqslant \phi_C(x) = \Pr(x \in S_U(C)) \leqslant \Pr(x \in \text{proj}_{2n+1}(S_U(B))) \ , \tag{3.45}$$

where B is given by membership function ϕ_B, where for any $V = (v_1, \ldots, v_{2n}, x)$,

$$\phi_B(v) \overset{df}{=} \min(\phi_{A_3}(h(v_{n+1}, \ldots, v_{2n})), \phi_{A_3}(j(v_1, \ldots, v_{2n}))\ \phi_{A_4}(v_1, \ldots, v_n, x))$$

$$= \begin{cases} \phi_D(v_1, \ldots, v_{2n}) \overset{df}{=} \min(\phi_{A_3}(h(v_{n+1}, \ldots, v_{2n})), \phi_{A_3}(j(v_1, \ldots, v_{2n})) \\ \qquad\qquad \text{iff } x = f(v_1, \ldots, v_{2n}) \\ \text{-------------------------} \\ 0 \qquad \text{iff } x \neq f(v_1, \ldots, v_{2n}) \ , \end{cases} \tag{3.46}$$

$$C \overset{df}{=} \text{proj}_{2n+1}(B) \tag{3.47}$$

is determined by, for all $x \in [0, 1]$,

$$\phi_C(x) = \sup_{\substack{\text{over all} \\ v_1, \ldots, v_{2n} \in [0,1]}} \phi_B(V) = \sup_{\left(\substack{\text{over all} \\ v_1, \ldots, v_{2n} \in [0,1] \\ \text{such that} \\ x = f(v_1, \ldots, v_{2n})}\right)} (\phi_B(v_1, \ldots, v_{2n}))$$

$$= \phi_{f(B)}(x) = \phi_{A_3}(\psi(x)) \ , \tag{3.48}$$

where

$$\psi(x) \overset{df}{=} \sup_{\underset{\sim}{0} \leqslant z \leqslant \underset{\sim}{1}} \ \sup_{\left(\substack{\underset{\sim}{0} \leqslant V', V'' \leqslant \underset{\sim}{1} \\ (1/n)\underset{\sim}{1}^T \cdot V' = z \\ (1/n)\underset{\sim}{1}^T \cdot V'' = x}\right)} \min(z, (1/nz)\ V'^T \cdot V'') \ ,$$

$$V', V'' \in \mathbb{R}^n \ , \ \underset{\sim}{0} \overset{df}{=} \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \ , \quad \underset{\sim}{1} \overset{df}{=} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} . \tag{3.49}$$

By a geometric argument, it can be shown $\psi(x) > x$. In particular, for $n = 2$,

$$\psi(x) = \begin{cases} \sqrt{x} \ , & \text{for } 0 \leqslant x \leqslant 1/2 \\ \psi_2(x), & \text{for } 1/2 \leqslant x \leqslant 1 \ , \end{cases}$$

where $\psi_2(x)$ is the solution z of

$$z = \frac{1 + (2z - 1)\ (2x - 1)}{2z} \ .$$

Thus, C can be interpreted as

$$C = \text{somewhat less than many (to the degree } \psi(x) \text{ exceeds x)} \ . \tag{3.50}$$

Equations (3.45), (3.48), and the monotone increasing property of ψ imply

$$x \geqslant \psi^{-1}(\phi_{A_3}^{-1}(\alpha)) \tag{3.51}$$

and

$$S_U(C) = [\psi^{-1}(\phi_{A_3}^{-1}(U)), 1] \ . \tag{3.52}$$

Conclusion 2 is a simpler result:

Replacing, where required, x by y, and f by k ,

$$\alpha \overset{df}{=} \min(\alpha_1, \alpha_2) \leqslant \phi_C(y) = \Pr(y \in S_U(C)) \leqslant \Pr(y \in \mathrm{proj}_{2n+1}(S_U(B)) \ , \tag{3.53}$$

where now for $V = (v_1, \ldots, v_{2n}, y)$,

$$\phi_B(v) = \begin{cases} \phi_D(v_1, \ldots, v_{2n}) \text{ iff } y = k(v_1, \ldots, v_{2n}) \\ 0 \quad \text{iff } y \neq k(v_1, \ldots, v_{2n}) \ . \end{cases} \tag{3.54}$$

$C = \mathrm{proj}_{2n+1}(B)$ is now given by

$$\phi_C(y) = \sup_{v_1, \ldots, v_{2n} \in [0,1]} \phi_B(v) = \phi_{k(B)}(y)$$

$$= \phi_{A_3}\Big(\sup_{\substack{\text{all } v' \in \mathbb{R}^{2n} \\ \text{such that} \\ y = h(v') \cdot j(v')}} (\min(h(v'), j(v'))) \Big) \ ,$$

since $k(v') = h(v') \cdot j(v)$,

$$= \phi_{A_3}\Big(\sup_{0 \leqslant h(v') \leqslant 1} \min(h(v'), y/h(v')) \Big) = \phi_{A_3}(\sqrt{y}) \ . \tag{3.55}$$

Thus

$$C = \text{more or less-most} \ , \tag{3.56}$$

$$y \geqslant (\phi_{A_3}^{-1}(\alpha))^2 \ , \tag{3.57}$$

and

$$S_U(C) = [(\phi_{A_3}^{-1}(U))^2, 1] \ . \tag{3.58}$$

The next example illustrates an application to mixed fuzzy and random statements. *Example* 3. (Related to Zadeh [13], problem 5, page 2)

Premise:

Individual A: 'It is not quite true Kati is very old.'

Individual B: 'It is not true she is young.'

Individual C: 'I believe she is most likely between 40 and 45 years of age.'

Source D: 'Based on dental records, the probability that she is between 44 and 47 years of age is (at least) 0.9.'

Conclusion: How old is Kati?

Define:

$$V = v_1 = \text{age (Kati)} \in \Omega_1 = \mathbb{R}^+ \tag{3.59}$$

and fuzzy sets

$$A_1 \overset{df}{=} \text{young} \ , \tag{3.60}$$

$$A_2 \overset{df}{=} \text{very old} = \text{very}^2 \text{ not young} \ , \tag{3.61}$$

$$A_3 \overset{df}{=} \text{true} \ , \tag{3.62}$$

$$A_4 \overset{df}{=} \text{not true} \ , \tag{3.63}$$

$$A_5 \overset{df}{=} \text{not quite true} = \text{not } \sqrt{\text{true}} \ , \tag{3.64}$$

$$A_6 \overset{df}{=} \text{most likely} \ , \tag{3.65}$$

$$A_7 \overset{df}{=} \text{ordinary set } [40,45] \ , \tag{3.66}$$

$$A_8 \overset{df}{=} \text{ordinary set } [44,47] \ , \tag{3.67}$$

with membership functions

$$\phi_{A_1} : \mathbb{R}^+ \rightarrow [0,1] \tag{3.68}$$

is a monotone decreasing function (from 1 to 0),

$$\phi_{A_2}(v_1) = (1 - \phi_{A_1}(v_1))^2 \ , \tag{3.69}$$

$$\phi_{A_3} : [0,1] \rightarrow [0, 1] \ , \tag{3.70}$$

is a monotone increasing function-sharply so near 1 (from 0 to 1) ,

$$\phi_{A_4}(x) = 1 - \phi_{A_3}(x) \ , \ x \in [0,1] \ , \tag{3.71}$$

$$\phi_{A_5}(x) = 1 - \phi_{A_3}^{1/2}(x) \ , \ x \in [0,1] \ , \tag{3.72}$$

$$\phi_{A_6}(x) = \max(\beta x \ , 1 - \beta) \ , \ x \in [0,1] \ , \tag{3.73}$$

where, e.g., $\beta = 2/3$. (For other approaches to modeling degrees of possibility, see Ref. 18, pp. 24-27.)

$$\phi_{A_7}(v_1) = \begin{cases} 1 \text{ iff } v_1 \in [40,45] \\ 0 \text{ iff } v_1 \notin [40,45] \ , \end{cases} \tag{3.74}$$

etc.

In addition, let T_1 be any random subset of $\mathbb{R}^+$ such that outcome A_8 is observed and

$$0.9 \leqslant \Pr(v_1 \in T_1 | v_1) \ , \tag{3.75}$$

for all possible v_1 .

Then let (see Eq. (3.2))

$$\phi_{A(T_1)}(v_1) = \begin{cases} 0.9, \text{ for all } v_1 \in A_8 \\ 0.1, \text{ for all } v_1 \notin A_8 \ . \end{cases} \tag{3.76}$$

Thus, the premise becomes here, for all v_1 of interest:

$$(\phi_{A_5}(\phi_{A_2}(v_1)) \geqslant \alpha_1) \ \& \ (\phi_{A_4}(\phi_{A_1}(v_1)) \geqslant \alpha_2)$$
$$\& \ (\phi_{A_6}(\phi_{A_7}(v_1)) \geqslant \alpha_3) \ \& (\phi_{A(T_1)}(v_1) \geqslant 0.9) \ , \tag{3.77}$$

for unspecified $1 \geqslant \alpha_1, \alpha_2, \alpha_3 \geqslant 0$.

Equation (3.3) then implies (no projection needed here), for all v_1 satisfying Eq. (3.77)

$$\alpha \overset{\text{df}}{=} \min(\alpha_1, \alpha_2, \alpha_3, 0.9) \leqslant \phi_B(v_1) = \Pr(v_1 \in S_U(C)) \ , \tag{3.78}$$

where,

$$\phi_B(v_1) = \min\left\{\phi_{A_5}(\phi_{A_2}(v_1)), \phi_{A_4}(\phi_{A_1}(v_1)) \ , \ \phi_{A_6}(\phi_{A_7}(v_1)), \phi_{A(T_1)}(v_1)\right\}$$
$$= \min\left\{ 1 - \phi_{A_3}^{1/2}((1-\phi_{A_1}(v_1))^2) \ , 1 - \phi_{A_3}(\phi_{A_1}(v_1)) \ , \right.$$
$$\left. \begin{Bmatrix} \beta, \text{ for } v_1 \in A_7 \\ 1-\beta, \text{ for } v_1 \notin A_7 \end{Bmatrix}, \begin{Bmatrix} 0.9, \text{ for } v_1 \in A_8 \\ 0.1, \text{ for } v_1 \notin A_8 \end{Bmatrix} \right\}$$
$$= \begin{cases} a & \text{iff} \quad v_1^*(a) \leqslant v_1 \leqslant v_1^{**}(a) \\ 1 - \phi_{A_3}(\phi_{A_1}(v_1)) & \text{iff} \quad v_1 \leqslant \min(v_0, v_1^*(a)) \\ 1 - \phi_{A_3}^{1/2}((1-\phi_{A_1}(v_1))^2) & \text{iff} \quad v_1 \geqslant \max(v_0, v_1^{**}(a)) \end{cases} \tag{3.79}$$

where

$$a \overset{\text{df}}{=} \begin{cases} a_1 \overset{\text{df}}{=} \min(1-\beta, 0.1) & \text{iff } v_1 < 40 \text{ or } v_1 > 47 \\ a_2 \overset{\text{df}}{=} \min(\beta, 0.1) & \text{iff } 40 \leqslant v_1 < 44 \\ a_4 \overset{\text{df}}{=} \min(\beta, 0.9) & \text{iff } 44 \leqslant v_1 \leqslant 45 \\ a_3 \overset{\text{df}}{=} \min(1-\beta, 0.9) & \text{iff } 45 < v_1 \leqslant 47 \end{cases} \ ; \tag{3.80}$$

$$\begin{cases} v_1^*(a) \overset{\text{df}}{=} \phi_{A_1}^{-1}(\phi_{A_3}^{-1}(1-a)), \\ v_1^{**}(a) \overset{\text{df}}{=} \phi_{A_1}^{-1}(1 - \sqrt{\phi_{A_3}^{-1}((1-a)^2)}) \ , \end{cases} \tag{3.81}$$

$$v_0 \overset{\text{df}}{=} \phi_{A_1}^{-1}(y_0) \ , \tag{3.82}$$

with y_0 uniquely determined from

$$\phi_{A_3}^2(y_0) = \phi_{A_3}((1-y_0)^2) \ . \tag{3.83}$$

Then, w.l.o.g., ordering the possible values of a as : $a_1 \leqslant a_2 \leqslant a_3 < a_4$, Eqs. (3.78) and (3.79) imply:

For $0 \leqslant \alpha \leqslant \min(y_0, a_1)$,

$$v_1 \in S_{(\alpha)} \overset{\text{df}}{=} [v_1^*(\alpha) \ , v_1^{**}(\alpha)] \ .$$

For $a_1 < \alpha \leqslant \min(y_0, a_2)$,

$$v_1 \in S_{(\alpha)} \overset{df}{=} [v_1^{*}(\alpha), v_1^{**}(\alpha)] \cap [40,47] .$$

For $a_2 < \alpha \leqslant \min(y_0, a_3)$,

$$v_1 \in S_{(\alpha)} \overset{df}{=} [v_1^{*}(\alpha), v_1^{**}(\alpha)] \cap [44,47] .$$

For $a_3 < \alpha \leqslant \min(y_0, \alpha_4)$,

$$v_1 \in S_{(\alpha)} \overset{df}{=} [v_1^{*}(\alpha), v_1^{**}(\alpha)] \cap [44,45] . \tag{3.84}$$

(For $\alpha > \min(y_0, a_4)$, v_1 is vacuous.)

Similarly, S_U (B) is determined for any value of U, $0 \leqslant U \leqslant \min(y_0, a_4)$, by simply replacing α by U everywhere in Eq. (3.84). Note that for $\min(y_0, \alpha_4) < U \leqslant 1$,

$$S_U(B) = \varnothing . \tag{3.85}$$

Example 4. *Generalized Modus Ponens*

Premise: From experience, a sensor operator knows that in monitoring a target, when attribute A_1 (e.g., maneuvering) is strongly present, then probably attribute A_2 (e.g., being aware of the operator's presence) is mildly present. A_1 is a function of χ^2, the observed statistical goodness of model fit, while A_2 is a function of q , an m-dimensional vector, representing various intelligence information concerning the target. On one particular occasion, the operator observes that A_1 is only moderately present.

Conclusion: What can be said in the latter occasion about the presence of A_2?

Define

$$v_1 \overset{df}{=} \chi^2 \in \mathbb{R}^+,\ v_2 \overset{df}{=} q \in \mathbb{R}^m ,\ \text{and } V \overset{df}{=} (v_1, v_2) \in \mathbb{R}^+ \times \mathbb{R}^m , \tag{3.86}$$

$$A_1 \overset{df}{=} \text{attribute } A_1 \text{ present} , \tag{3.87}$$

$$A_2 \overset{df}{=} \text{attribute } A_2 \text{ present} , \tag{3.88}$$

$$A_3 \overset{df}{=} \text{attribute } A_1 \text{ very strongly present} , \tag{3.89}$$

$$A_4 \overset{df}{=} \text{attribute } A_2 \text{ mildly present} , \tag{3.90}$$

$$A_5 \overset{df}{=} \text{probably}, \tag{3.91}$$

a fuzzy subset of [0,1] ,

$$A_6 \overset{df}{=} \text{probably attribute } A_2 \text{ mildly present} , \tag{3.92}$$

$$A_7 \overset{df}{=} \text{if( ) then ( )}$$

$$= \Rrightarrow , \tag{3.93}$$

$$A_8 \overset{df}{=} \text{if } A_3 \text{ then } A_6 \tag{3.94}$$

$$= (A_3 \times \mathbb{R}^m) \Rrightarrow (\mathbb{R}^+ \times A_6) ,$$

a fuzzy subset of $\mathbb{R}^+ \times \mathbb{R}^m$.

The premise may then be stated as

$$(\phi_{A_8}(V) \geqslant \alpha_1) \ \& \ (\phi_{A_1}(V_1) \geqslant \alpha_2) \quad , \tag{3.95}$$

for $1 \geqslant \alpha_1, \alpha_2 \geqslant 0$, unspecified constants.

Assume that $\phi_{A_1}: \mathbb{R}^+ \to [0,1]$ is *onto* and it is reasonable to define, for all $V \in \mathbb{R}^+ \times \mathbb{R}^m$,

$$\phi_{A_1}(V) \overset{\text{df}}{=} \phi_{A_1}(v_1) \quad , \quad \phi_{A_2}(V) \overset{\text{df}}{=} \phi_{A_2}(v_2) \quad ,$$

$$\phi_{A_3}(V) \overset{\text{df}}{=} \phi_{A_3}(v_1) \overset{\text{df}}{=} \phi^3_{A_1}(v_1) \quad ; \tag{3.96}$$

$$\phi_{A_4}(V) \overset{\text{df}}{=} \phi_{A_4}(v_2) \overset{\text{df}}{=} \phi^{1/2}_{A_2}(v_2) \quad . \tag{3.97}$$

From Eq. (3.73) it follows, for β fixed at say ½ ,

$$\phi_{A_5}(x) \overset{\text{df}}{=} \max(\beta x \, , \, 1-\beta), \tag{3.98}$$

for all $x \in [0,1]$.

Thus,

$$\phi_{A_6}(V) = \phi_{A_6}(v_2) = \phi_{A_5}(\phi_{A_4}(v_2)) \quad , \tag{3.99}$$

$$\phi_{A_7}(x, y) \overset{\text{df}}{=} \max(1-x, y) \quad , \tag{3.100}$$

for all $x, y \in [0,1]$. (See Eq. (2.22).)

Hence, for all $V \in \mathbb{R}^+ \times \mathbb{R}^m$,

$$\phi_{A_8}(v) = \phi_{A_7}(\phi_{A_3}(v_1), \phi_{A_6}(v_2)) \quad . \tag{3.101}$$

Then Eq. (3.6) implies for all v_1 satisfying (3.95),

$$\begin{aligned} \alpha = \min(\alpha_1, \alpha_2) \leqslant \phi_C(v_2) &= \Pr(v_2 \in S_U(C)) \\ &\leqslant \Pr(v_2 \in \mathrm{proj}_2(S_U(B))) \quad , \end{aligned} \tag{3.102}$$

where

$$C \overset{\text{df}}{=} \mathrm{proj}_2(B) \quad , \tag{3.103}$$

$$B \overset{\text{df}}{=} A_8 \cap A_1 \quad , \tag{3.104}$$

and for all $V \in \mathbb{R}^+ \times \mathbb{R}^m$,

$$\phi_B(V) = \min(\max(1-\phi^3_{A_1}(v_1), \beta \cdot \phi^{1/2}_{A_2}(v_2) \, , 1-\beta) \, , \phi_{A_1}(v_1)) \tag{3.105}$$

$$= \begin{cases} \phi_{A_1}(v_1) \text{ iff } 0 \leqslant \phi_{A_1}(v_1) \leqslant \max(x_0 \, , \phi_{A_6}(v_2)) \\ 1 - \phi^3_{A_1}(v_1) \text{ iff } x_0 \leqslant \phi_{A_1}(v_1) \leqslant (1-\phi_{A_6}(v_2))^{1/3} \\ \phi_{A_6}(v_2) \text{ iff } \max((1-\phi_{A_6}(v_2))^{1/3}, \phi_{A_6}(v_2)) \leqslant \phi_{A_1}(v_1) \leqslant 1. \end{cases}$$

By graphical inspection ,

$$\phi_C(v_2) = \sup_{v_1 \in \mathbb{R}^+} \phi_B(v) = \max(x_0 \, , \phi_{A_6}(v_2)) \quad , \tag{3.106}$$

where

$$\phi_{A_6}(v_2) = \max\,(\beta \cdot \phi^{1/2}_{A_2}(v_2)\ , 1-\beta)\ , \tag{3.107}$$

and x_0 uniquely satisfies the equation

$$x_0 = 1 - x_0^3\ . \tag{3.108}$$

Thus, Eqs. (3.102) and (3.106) imply that v_2 is such that:

If $0 \leqslant \alpha \leqslant \gamma(\beta)$, then $v_2 \in \mathbb{R}^m$ can be arbitrary. (3.109)

If $\gamma(\beta) < \alpha \leqslant 1$, then

$$(\alpha/\beta)^2 \leqslant \phi_{A_2}(v_2)\ , \tag{3.110}$$

where

$$\gamma(\beta) \overset{\mathrm{df}}{=} \max\,(1-\beta, x_0)\ . \tag{3.111}$$

Hence,

$$S_U\,(C) = \begin{cases} \mathbb{R}^m\ , \text{ iff } 0 \leqslant U \leqslant \gamma(\beta) \\ \phi^{-1}_{A_2}\,([(U/\beta)^2, 1])\ , \text{ iff } \gamma(\beta) < U \leqslant 1\ . \end{cases} \tag{3.112}$$

Here, C may be interpreted from Eq. (3.106) to mean

C = A_2 is somewhat less than (to the degree that max $(x_0\ , \phi_{A_6}(v_2))$ exceeds $\phi_{A_6}\,(v_2)$) probably mildly present.

CONCLUDING REMARKS

A relatively simple tie-in has been established between all fuzzy and random subsets of a given space, via the canonical mapping of Eqs. (2.3) and (2.4). In the previous section, it was shown that this mapping relates fuzzy and probabilistic reasoning through the establishment of equivalent fuzzy and random confidence sets with respect to the unknown variable. Connections between fuzzy set techniques and probabilistic methods remain to be explored in other fields of application, such as in fuzzy topology (e.g. [27] and [28]), fuzzy decision theory (e.g. [1]), or fuzzy clustering and partitioning (e.g. [2,3,29]). For example, in the latter, the most common constraint on a fuzzy partitioning $\{\phi_{A_1}, \ldots, \phi_{A_k}\}$ of a set X is to require, for all $x \in X$,

$$\sum_{i=1}^{k} \phi_{A_i}(x) = 1\ . \tag{4.1}$$

However, applying Eqs. (2.10) and (2.19) to Eq. (4.1), it follows that for all $x \in X$,

$$\Pr\,(x \in S_U\,(\bigcup_{i=1}^{k} A_i)) = \Pr\,(x \in \bigcup_{i=1}^{k} S_U\,(A_i)) \leqslant 1\ , \tag{4.2}$$

with strict inequality holding in general.

Thus, perhaps, a more natural constraint (but possibly less computationally feasible) is to require $\Pr\,(x \in S_U\,(\bigcup_{i=1}^{k} A_i)) = 1$, i.e.,

$$\max_{i=1,\ldots,k} \phi_{A_i}(x) = 1 \ , \tag{4.3}$$

so that all points in X are guaranteed coverage with probability one.

The following problems and results warrant further investigation:

(1) Results (1.1) and (2.10) show that for any given space, the collection of all random subsets is decomposed into equivalence classes, where each such class is equivalent w.r.t. one point coverages to a particular fuzzy subset. One representative of any such class is the random set produced by the canonical map (Eq. (2.4)).

What direct characterizations exist for these equivalence classes?

What functions, other than the canonical one, can be explicitly constructed mapping each fuzzy subset into some representative of the corresponding equivalence class of random sets? Are any of these functions isomorphic-like in structure with respect to standard fuzzy and random set operations?

(2) Determine mappings from the class of all multiple point coverage functions of some prescribed type into the class of all random subsets, for a given space, which preserve these coverages. For example, let X be a given space and $G \overset{\text{df}}{=}$ the set of all two point coverage functions, i.e., for any typical $\phi \in G$, $\phi:\{\{x, y\} | x, y \in X\} \rightarrow [0,1]$, with the restriction $\phi(\{x, y\}) \leqslant \min(\phi(x), \phi(y))$, where $\phi(x) \overset{\text{df}}{=} \phi(\{x,x\})$, etc. Then find a computable K: $G \rightarrow R$ such that for all x, y $\in$ X,

$$\phi(\{x,y\}) = \Pr(\{x,y\} \subseteq K(\phi)) \ . \tag{4.4}$$

(3) Some partial responses to the questions posed above in (1) and (2).

(i) A general construction might be established which determines a probability measure over a set of subsets of a given space X, and hence, in a sense, a random subset of X, which is equivalent to a given fuzzy subset of X for one point coverages. A similar result may hold for given multiple point coverage functions:

Let X be a given space with σ-ring of subsets B (X). Extending Eq. (2.11), for any C $\in$ B (X) define

$$C_C \overset{\text{df}}{=} \{B | B \in B(X) \ \& \ B \supset C\} \ . \tag{4.5}$$

(I) Suppose, fuzzy subset A of X is given. Then, first define, for any x $\in$ X ,

$$\nu_A(C_{\{x\}}) \overset{\text{df}}{=} \phi_A(x) \ . \tag{4.6}$$

Then define, inductively, for distinct $x_1, \ldots, x_m \in X$, and each integer $m \geqslant 2$

$$0 < \nu_A(C_{\{x_1,\ldots,x_m\}}) \leqslant \min_{\substack{\text{over all} \\ 1 \leqslant j_1 < \cdots < j_{m-1} \leqslant m}} \nu_A(C_{\{x_{j_1},\ldots,x_{j_{m-1}}\}}) \ . \tag{4.7}$$

For any finite sets $C_1, C_2 \in B(X)$, define

$$\nu_A(C_{C_1} \dashv C_{C_1 \cup C_2}) \overset{df}{=} \nu_A(C_{C_1}) - \nu_A(C_{C_1 \cup C_2}) ,$$

$$\nu_A(C_{C_1} \cup C_{C_2}) \overset{df}{=} \nu_A(C_{C_1}) + \nu_A(C_{C_2}) - \nu_A(C_{C_1 \cup C_2}) , \tag{4.8}$$

etc.

It can be seen that the ring of sets, R, generated by the $C_{\{x\}}$'s consists of all finite unions of sets of the form $C_C \dashv C_{\{x_1\}} \dashv \cdots \dashv C_{\{x_n\}}$ where C is finite and $C \cap \{x_1, x_2, \ldots, x_n\} = \varnothing$.

Then if, e.g., continuity from above at $\varnothing$ could be shown [30]—possibly using the facts that $\nu_A(C_C) \leqslant \mu_{\Gamma_A}(C_C)$ (from Eqs. (2.18) and (4.7)) and $\nu(C_{\{x\}}) = \mu_{\Gamma_A}(C_{\{x\}})$, and the continuity of μ_{Γ_A} at ϕ as a probability measure—then ν_A would be a probability measure on R.

Then, using standard results (e.g., Ref. 30, pp. 41-62), ν_A could be extended uniquely to a complete probability measure over $\overline{\sigma}(R)$, the σ-ring of all outer ν_A-measurable sets.

Thus, the probability space $(B(X), \overline{\sigma}(R), \nu_A)$ would be identified with a 'random set' over $B(X)$ which coincides with ϕ_A with respect to all one point coverages via Eq. (4.6). (See (2.10), R.H.S.)

(II) Similarly, for unrestricted multiple point coverages, begin the construction with Eq. (4.6) replaced by

$$\nu_A(C_C) \overset{df}{=} \phi(C) , \tag{4.9}$$

where $\phi : B(X) \to [0, 1]$ is a given coverage function with the required constraint

$$\phi(B) \geqslant \phi(C) , \tag{4.10}$$

for all $\varnothing \neq B \subseteq C$.

Then, by carrying out the remainder of the construction as in part (I) for fuzzy sets, a probability space might be developed which is identifiable with a random set that is equivalent to ϕ, with respect to all set coverages.

Note that for ν_A in (I), the $C_{\{x\}}$'s essentially generate the σ-ring $\overline{\sigma}(R)$, whereas for μ_{Γ_A} (Eqs. (2.7)-(2.11)) the $C_{\{x\}}$'s are merely elements of the σ-ring $\Gamma_A(B_1)$. It should also be pointed out that $\overline{\sigma}(R)$ and $\Gamma_A(B_1)$ are relatively sparse compared to σ-ring $B(B(X))$ for general random sets over $B(X)$.

(ii) Any probability measure ν_A over $\overline{\sigma}(R)$ which is absolutely continuous w.r.t. μ_{Γ_A} (and equivalent w.r.t. all one point coverages), is characterized by a r.v. V over [0, 1] having p.d.f. h determining probability measure μ over B_1 such that

$$\nu_A = \mu(\Gamma_A^{-1}) , \tag{4.11}$$

equivalently, ν_A corresponds to random set

$$S \overset{df}{=} \Gamma_A(V) , \tag{4.12}$$

and h satisfies

$$t = \int_{s=0}^{t} h(s)\, ds \ ; \ \text{for all } t \in \operatorname{rng}(\phi_A) . \tag{4.13}$$

Clearly, $h \equiv 1$ satisfies Eq. (4.13), yielding back, $S = S_U(A)$, U uniform over [0,1]. Also, by differentiating Eq. (4.13) with respect to t, $h(t) = 1$, for all t in any interval that exists which is a subset of rng (ϕ_A). On the other hand, if rng (ϕ_A) is discrete, say $0 = y_0 < y_1 < \cdots < y_n = 1$, then the solution set of h's satisfying Eq. (4.13) is determined by

$$h(s) \overset{df}{=} f_j(s) \cdot (y_j - y_{j-1}) \Big/ \int_{t=y_{j-1}}^{y_j} f_j(t)dt \quad , \tag{4.14}$$

for all $s \in (y_{j-1}, y_j]$; $j = 1, 2, .., n-1$, where each f_j is arbitrary nonnegative having a positive integral over $(y_{j-1}, y_j]$.

(iii) A wide variety of integral and differential equations can be obtained which characterize various subclasses of solutions to the one or multiple point coverage problem.

For example, let A be any given fuzzy subset of $\mathbb{R}$ with ϕ_A differentiable everywhere and unimodal about some x_0 with maximal value $1 > \alpha_0 = \phi_A(x_0) > 0$, and such that $\lim_{|s|\to\infty} \phi_A(s) = 0$. Consider then the class of all random intervals S of the form $(W - a, W + a)$, where $a > 0$ is constant and W is a r.v. over $\mathbb{R}$ having some p.d.f. h. Then it is desired to determine those a's and h's for which S and A are equivalent w.r.t. one point coverages:

The characterizing equation is

$$\int_{s=x-a}^{x+a} h(s)\, ds = \phi_A(x) \; ; \; \text{all } x \in \mathbb{R} \quad , \tag{4.15}$$

yielding by differentiation

$$h(x + a) - h(x - a) = \frac{d\phi_A(x)}{dx} \; ; \; \text{all } x \in \mathbb{R} \; . \tag{4.16}$$

In turn, Eqs. (4.15) and (4.16) yield the solution set consisting of all a's satisfying

$$1 = \sum_{j=-\infty}^{+\infty} \phi_A(x_0 + (2j-1) \cdot a) \quad , \tag{4.17}$$

and all h's of the form

$$h(x \pm 2ka) = h(x) \pm \sum_{j=1}^{k} \frac{d\phi_A(x \pm (2j-1) \cdot a)}{dx} \quad , \tag{4.18}$$

for all $x \in [x_0, x_0 + 2a)$, $k = 1, 2, \ldots$; with h arbitrary nonnegative over $[x_0, x_0 + 2a)$ such that Eq. (4.15) is satisfied at $x = x_0 + a$.

(iv) Let ϕ be any given two-point coverage function over X as discussed in subsection (2) above, where, without loss of generality, $\phi : X \times X \to [0,1]$, the membership function of some symmetric fuzzy subset A of X × X, i.e., for all x, y ∈ X, $\phi(x, y) = \phi(y, x) = \phi(\{x, y\})$.

In general, any attempt at using formally $S_U(A) \overset{df}{=} \phi^{-1}[U,1]$, the analogue of Eq. (2.4) for one point coverages, will not yield back equivalence between $S_U(A)$ and ϕ w.r.t. two point coverages.

Necessary and sufficient conditions for equivalence to occur are (using Eq. (2.32))

$$S_U(A) = \text{proj}_1\,(S_U(A)) \times \text{proj}_1\,(S_U(A)) \;; \tag{4.19}$$

equivalently,

$$\phi(x, y) \geqslant u \text{ and } \phi(y, z) \geqslant u \tag{4.19'}$$

implies $\phi(x, z) \geqslant u$; for all $x, y, z \in X$, and all $u \in [0, 1]$.

For an example of a ϕ satisfying the above conditions, define for all $x, y \in X$,

$$\phi(x, y) \stackrel{\text{df}}{=} f(\min(\phi(x), \phi(y))) \;, \tag{4.20}$$

where $f: [0, 1] \to [0, 1]$ is any fixed monotone increasing function such that for all $t \in [0, 1]$, $f(t) \leqslant t$. For example, $f(t) \stackrel{\text{df}}{=} \alpha t$, $0 < \alpha \leqslant 1$. Note, for $\alpha = 1$, (4.20) is a special case of Eq. (2.18) for $C = \{x, y\}$, when $\phi(x, y) \stackrel{\text{df}}{=} \Pr(\{x, y\} \subseteq S_U(A))$, for any given fuzzy subset A of X .

(v) If X is a discrete space, say, $X = \{x_1, .., x_m\}$, then the one point coverage problem reduces to solving a system of $(m + 1)$ linear equations—with coefficients 0 or 1—in 2^m unknowns:

Define random set S to be determined equivalent to given fuzzy subset A of X by,

$$\Pr(S = B) \stackrel{\text{df}}{=} a_B \;, \tag{4.21}$$

where $B \subseteq X$, and the 2^m a_B's are determined from

$$\begin{cases} \phi_A(x_j) = \Pr(x_j \in S) = \sum\limits_{\left(\substack{\text{over all}\\ B\text{, with}\\ x_j \in B \subseteq X}\right)} a_B \;; \quad j = 1, 2, \ldots, m \\ 0 \leqslant a_B \leqslant 1, \text{ for all } B \subseteq X \;; \quad \sum\limits_{B \subseteq X} a_B = 1 \;. \end{cases} \tag{4.22}$$

One solution of Eq. (4.22) is furnished by $S = S_U(A)$, yielding

$$S_A(U) = B_j \stackrel{\text{df}}{=} X \dashv \phi_A^{-1}(y_0) \dashv \cdots \dashv \phi_A^{-1}(y_j) \;, \tag{4.23}$$

for all $U \in (y_j, y_{j+1}]$; $j = 0, 1, .., n-1$, where $\text{rng}(\phi_A)$ is ordered as $0 = y_0 < y_1 < \cdots < y_n = 1$, where possibly y_0 and /or y_n are vacuous.

In this case, it follows for any $B \subseteq X$,

$$a_B = \begin{cases} 0 & \text{iff } B \neq B_j \,, \text{ for any } j \\ y_{j+1} - y_j & \text{iff } B = B_j \,, \text{ for some } j \;. \end{cases} \tag{4.24}$$

Similarly, for any k , $m - 1 \geqslant k \geqslant 1$, the k point coverage problem for X requires the solution of $\sum\limits_{j=0}^{k} \binom{m}{j}$ linear equations—with coefficients 0 or 1—in 2^m unknowns.

Simple characterizations of solutions to these equations would be useful.

(4) What about the converse of the thrust of this paper? Can fuzzy set theory—or perhaps its generalization, as given above, by multiple point coverage functions, be used to determine, in some sense, random set theory? There may be some relation here with the concept of 'traps', as presented by Kendal [31] (Equation (2.17) can be reinterpreted in terms of trapping functions), or perhaps with Shafer's belief functions [17].

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